Quantitative continuity and computable analysis in Coq

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1. Represented spaces, realizers and continuity in incone
2. Metric spaces and continuity
3. Closed choice on the natural numbers

Represented spaces

Represented space: $\mathbf{X} = (X, \delta_{\mathbf{X}})$, where $\delta_{\mathbf{X}} :\subseteq \mathcal{B} \to X$.
Here $\mathcal{B} = \mathbb{N}^{\mathbb{N}}$ or $\mathcal{B} = \mathbf{A}^{\mathbf{Q}}$ and $\mathbf{Q}$, $\mathbf{A}$ sufficiently concrete.
if $\delta(\varphi) = x$ then φ is **description** or **name** of x.

Example (The Cauchy reals $\mathbb{R}_{\mathbb{Q}}$)

Use $\mathbf{Q} := \mathbb{Q} =: \mathbf{A}$ and $\delta_{\mathbb{R}_{\mathbb{Q}}} :\subseteq \mathcal{B} \to X$ specified by

$$\varphi\colon \mathbb{Q} \to \mathbb{Q} \text{ is description of } x \iff \forall \varepsilon > 0, |x - \varphi(\varepsilon)| \leq \varepsilon.$$

$x \in \mathbf{X}$ is computable if it has computable name.
π, e are computable.
Topology on $\mathbf{X}$: push forward topology on $\mathcal{B}$.

Represented spaces in incone

Definition (Continuity space)

Record consisting of:

X: abstract type to compute over.

$\mathbf{Q}$, $\mathbf{A}$: countable, inhabited types.

δ: deterministic, surjective relation $\mathbf{A}^{\mathbf{Q}} \to X \to \texttt{Prop}$.

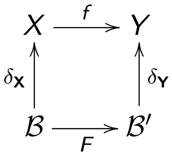
Example (The Cauchy Reals RQ)

$\mathbb{R}_{\mathbb{Q}} \rightsquigarrow$ continuity space RQ. Using types from standard library.

```
rep_RQ:= make_mf ((phi: Q -> Q) (x: R) =>
∀eps, 0 < eps -> Rabs (x - Q2R (phi eps)) <= eps).
```

Continuity and computability of functions

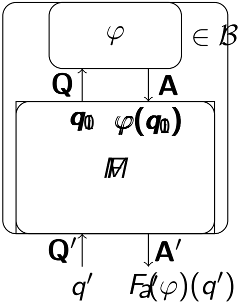
$F :\subseteq \mathcal{B} \to \mathcal{B}'$ **realizes** $f : \mathbf{X} \to \mathbf{Y}$ if
φ name of $x \rightsquigarrow F(\varphi)$ name of $f(x)$.
$\delta_{\mathbf{Y}} \circ F$ extends $f \circ \delta_{\mathbf{X}}$.
continuous: has continuous realizer.
computable: computable realizer.



Example

Cauchy reals $\rightsquigarrow$ ε-δ-continuity.

Computability is what you know (?) from computable analysis.
Computability implies continuity.



Continuity in Coq

Definition (continuity on Baire space)

$F :\subseteq \mathbf{A}^{\mathbf{Q}} \to \mathbf{A}'^{\mathbf{Q}'}$ is continuous if for all $\varphi \in \mathrm{dom}(F)$ and $q' \in \mathbf{Q}'$ there exists a finite list $K \subseteq \mathbf{Q}$ such that for all $\psi \in \mathrm{dom}(F)$

$$\varphi|_K = \psi|_K \implies F(\varphi)(q') = F(\psi)(q').$$

Partiality uses deterministic relations or dependent types.
(q_n) enumeration of $\mathbf{Q}$ $\rightsquigarrow$ metric on $\mathcal{B} = \mathbf{A}^{\mathbf{Q}}$.
$d_{\mathcal{B}}(\varphi, \psi) := 2^{-\min\{n|\varphi(q_n)\neq\psi(q_n)\}}$.

Lemma (`lim_lim`)

convergence relation wrt. $d_{\mathcal{B}}$ is pointwise convergence.

Lemma (`cont_cont`)

$F :\subseteq \mathcal{B} \to \mathcal{B}$ is continuous iff it is ε-δ-continuous wrt. $d_{\mathcal{B}}$.

the metric representation

(M, d) metric space (r_i) dense sequence in M

$$\delta_{\mathbf{M}}(\varphi) = x \iff \forall n, d(x, r_{\varphi(n)}) \leq 2^{-n}.$$

$\mathbf{M} := (M, \delta_{\mathbf{M}})$ is a represented space.

Lemma (cont_mcont, mcont_cont)

$f : \mathbf{M} \to \mathbf{N}$ is continuous iff it is ε-δ-continuous.

Lemma (lim_lim)

Metric convergence relation: exists convergent sequence of names.

Lemma (rlzr_scnt)

f sequentially continuous iff it has sequentially continuous realizer.

Lemma (cont_scnt)

$f : \mathbf{M} \to \mathbf{N}$ is continuous iff it is sequentially continuous.

Sequences and the real numbers

Lemma (Rplus_cont, Rmult_cont)

Addition and multiplication of reals is continuous (computable).

$\mathbf{X}$ represented space $\rightsquigarrow$ represented space $\mathbf{X}^\omega$.
questions: $\mathbb{N} \times \mathbf{Q}$; answers: $\mathbf{A}$.
i.e. names are functions of type $\mathbb{N} \times \mathbf{Q} \to \mathbf{A}$.

$$\delta_{\mathbf{X}^\mathbb{N}}(\varphi) = (x_n) \iff \forall n, \delta_{\mathbf{X}}(\varphi(n, \cdot)) = x.$$

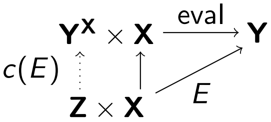
Example (lim_not_cont, lim_eff_cont)

Consider $\lim\colon \subseteq \mathbb{R}^\mathbb{N} \to \mathbb{R}$. Is discontinuous. Restriction to sequences with $|x_n - x_m| \leq 2^{-n} + 2^{-m}$ is continuous (computable).

Function spaces

For each $\mathbf{X}$ and $\mathbf{Y}$ there is a represented space $\mathbf{Y}^{\mathbf{X}}$ such that:

- evaluation is continuouscomputable.
- currying is possible (computably).
- the underlying set are the continuous functions.
- $f \in \mathbf{Y}^{\mathbf{X}}$ has computable name iff it has computable realizer.



Standard construction: Weihrauch's η (for TTE).
Implemented in incone. But uses a slightly different construction.
The realizer for evaluation is inherently partial.

discrete spaces

For countable D set $\mathbf{Q} := \{\star\}$, $\mathbf{A} := D$ and

$$\delta_{\mathbf{D}}(\varphi) = x \iff \varphi(\star) = x.$$

Lemma (rep_id_dscrt)

For rep. space $\mathbf{D} := (D, \delta_{\mathbf{D}})$ any $f : \mathbf{D} \to \mathbf{X}$ is continuous.

$\rightsquigarrow$ $\mathbb{N}$ is discrete space and $\mathbf{X}^{\mathbb{N}}$ are all sequences from $\mathbf{X}$.
Earlier we used $\mathbf{X}^{\omega}$ as the space of all sequences.

Lemma (cprd_is_fun)

$\mathbf{X}^{\mathbb{N}}$ is (computably) isomorphic to $\mathbf{X}^{\omega}$, i.e. $\mathbf{X}^{\mathbb{N}} \simeq \mathbf{X}^{\omega}$.

Proof.

One direction is simple, other uses the realizer for evaluation. □

Sirpinski space and hyperspaces

$C_{\mathbf{X}} \colon \mathcal{A}(\mathbf{X}) \rightrightarrows \mathbf{X}, A \mapsto A$. Used for classifying incomputability.
$\mathcal{A}(\mathbf{X})$: space of closed sets given as complements of opens.
$\mathcal{O}(\mathbf{X}) = \Sigma^{\mathbf{X}}$. Where $\Sigma = (\{\top, \bot\}, \delta_\Sigma)$ with $\mathbf{Q} := \mathbb{N}$, $\mathbf{A} = \mathbb{B}$ and

$$\delta_\Sigma(\varphi) = \bot \iff \forall n, \varphi(n) = \mathtt{false}.$$

If $\mathbf{X} = \mathbf{M}$, then the elements of $\mathcal{O}(\mathbf{M})$ are exactly the open sets.

Lemma (ON_iso_Onat, AN_iso_Anat)

$\mathcal{O}(\mathbb{N})$ and $\mathcal{A}(\mathbb{N})$ encode via enumeration.

Proof.

For opens: Concrete space $\mathcal{O}_\mathbb{N}$, then $\mathcal{O}(\mathbb{N}) = \Sigma^\mathbb{N} \simeq \Sigma^\omega \simeq \mathcal{O}_\mathbb{N}$. □

Lemma

$C_\mathbb{N} \colon \mathcal{A}(\mathbb{N}) \rightrightarrows \mathbb{N}$ *is discontinuous.*

Conclusion

1. incone allows to formalize results from computable analysis.
2. broad scope (real analysis, synthetic descriptive set theory).
3. check out incone (project homepage).
4. extended version of the paper on hal.

What's next?

1. computable Weierstraß approximation theorem ($\mathbb{R}^{[0,1]} \simeq C([0,1])$).
2. use formalization of model of computation.
3. extraction of continuity information.
4. more real analysis, efficient algorithms, code extraction.

Thanks.